Biology

Golden ratio (*Sectio Aurea*) in the Elliptical Honeycomb

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The honeybee comb, which is highly similar among honeybee species, is a mass of six–sided cells made by honeybees. It contains the brood, the honey and the pollen within horizontally–arranged and parallel structures. The construction processes and the geometry of the hexagonal cells have been extensively studied since centuries. Although studies of the natural, full–sized comb structure have been thoroughly performed in the past, the analysis of their early developmental stage size properties has not been investigated in detail. Here in particular, I found that the general two–dimensional elliptical form of the newly–constructed honeycombs could be drawn into a rectangle of modules having values approaching either 2.00 or 1.62, where the module of the rectangle is the simple division of it's long by it's short side lengths. It is proposed here that the elliptical form of the early stage honeybee comb is not random, but is following mathematical rules reflecting some geometry intimately related to the golden ratio, also called golden mean or divine proportion. This mathematical presence of the golden ratio might reveal the effect of an inherent law of the Cosmos in the honeybees' world.

Apis mellifera | honeybee | honeycomb | ellipse | golden ratio | gnomon

Introduction

A honeycomb is certainly one of the most beautiful and impressive natural structures. It's study is not only allowing the understanding of the honeybees' natural conditions of living, but it is also a fascinating piece of natural wax architecture. The honeycomb is a crucial part of the honeybee's nest, whose composition, structure and function have been extensively reviewed [1]. An exhaustive coverage of the synthesis and secretion of beeswax, its elaboration into combs and the factors that bear on the execution of these processes by honeybees has been reviewed elsewhere [2]. In particular, the construction of the hexagonal cells and the regulation of the space between adjacent combs have been a matter of extensive research [3]. Analyses of freely–built combs in mixed or pure *A. mellifera* and *A. cerana* (Hymenoptera, Apoidae, Apidae) colonies, in terms of numbers of festoons, number of honeybee workers on festoons, percentage of irregular cells, cell size and patterns of newly built combs, have been presented elsewhere [4]. The biological foundations of swarm intelligence of bees, ants and locusts have been extensively reviewed ; in the case of the honeybee, the pattern of the hexagonal cells in the combs can be thought to be the result of the darwinian natural selection, or the application of simple physical or mathematical rules [5]. The formation of the hexagonal pattern itself can be explained by wax flowing in liquid equilibrium, in which "the structure of the combs of honeybees results from wax as a thermoplastic building medium, which softens and hardens as a result of increasing and decreasing temperatures" [6]. These original results were supported later by independent researchers [7].

Physicians have proposed a mathematical model for honeycomb construction in which, via a set of dynamical coupled partial differential equations, the essential dynamical features of bee–bee and bee–wax interactions are integrated [8]. This domain has been the subject of intensive research (reviewed in [9]). Two main mechanisms can hypothetically explain the hidden geometry of the honeycomb, namely the diffusion-limited aggregation [10], and the constructal theory [11].

I have certainly not the perception that mathematics is a dull subject with no connection to real life. As a scientist, I am also not " studying nature just because it is useful, but because it is beautiful" [12]. Among many other scientists, I believe mathematics is nature's language. Sometimes this language is (very) complicated. Sometimes it is more simple, but not directly evident, or even hidden.

It is obvious that honeycombs do present, at peculiar stages during their growth progression, the signature or pattern of ellipses (Figure 1). This is a well–known and widely accepted fact, since the normal sketch of the comb does possess an ellipsoidal form that is typically described by all of the authors involved in the study of the beeswax construction [13,14].

In this article, I am presenting circumstantial evidence that the elliptical honeycomb is based on the golden ratio. The golden ratio is an irrational number. It is represented by the Greek letter Φ or ϕ (Phi) and has the value 1.6180339887, approximately [15]. The value of Φ is calculated as $(1 + \sqrt{5})$ divided by 2. Throughout history, the golden ratio has been studied not solely by mathematicians and philosophers, but also by biologists, naturalists, artists, architects and musicians, since it was also for them an essential element for the creation and keeping of order, form and beauty [16,17]. The fascinating presence of the golden ratio in the early honeycomb is an additional stone in the edification of the Cosmos, in which it's ubiquitous presence can only be deciphered but not formally explained.

Methods

The experiments were conducted in a public land owned by the community of the village of Aclens (Vaud, Switzerland), with the informed consent of the legal authorities. The regular permission to perform beekeeping at this location was obtained from the Service de la Consommation et des Affaires Vétérinaires (SCAV ; Lausanne, Switzerland) du Canton de Vaud (beekeeper number : 5004/5621, since 2008). No specific permissions were required to perform the field experiments at this location, since beekeeping is regularly

Conflict of interest: No conflicts declared
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performed in this area. These field studies did not involve endangered or protected species. Eight healthy colonies (in Dadant–Blatt hives) were employed for honeycomb analyses that were performed between March 2015 and July 2015. In each hive, two to three frames out of a total of eleven consisted of so–called "foundationless" frames [18] allowing close–to–natural construction of the combs by the honeybees. As an apiary adviser in the Canton de Vaud (Switzerland), I followed the official beekeeping procedures provided by the Bee Research Center at the Agroscope Liebefeld–Posieux (Berne, Switzerland) [19]. During the previous autumns and winters, the *Apis mellifera carnica* honeybees have been treated against the varroa mite *Varroa destructor* with acetic acid and oxalic acid, essentially as recommended elsewhere [20]. Bee hives were located 5 km north of the city of Morges (altitude 510 m above sea level; Switzerland). In the geographic area where the study took place, the bees usually begin foraging to collect nectar and pollen in early March, depending on the weather conditions. The hives were opened no more than once every seven to ten days, in order to keep the honeybees quiet most of the time. The presence of a laying queen and brood was regularly controlled. The observations of the frames and the honeycombs were performed following established and regular beekeeping practices, and did not harm the honeybees.

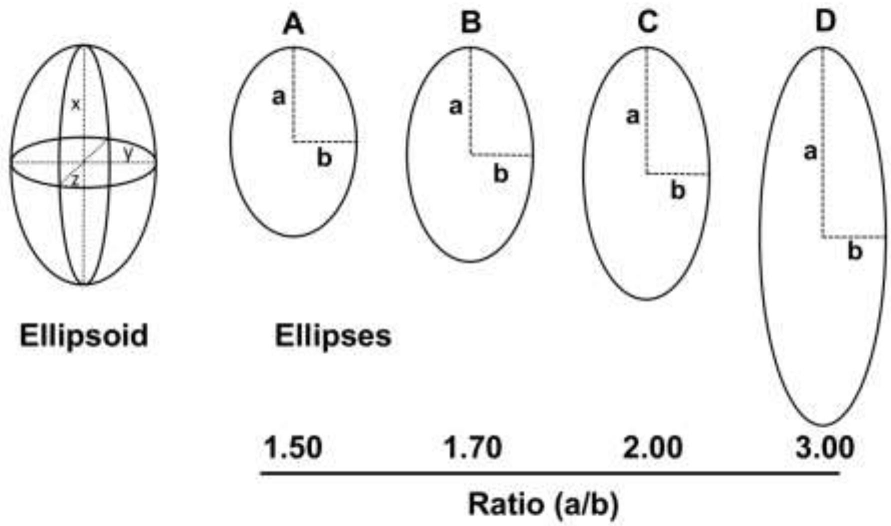

Figure 1. Some characteristics of the ellipsoid and the ellipses. Ellipsoid is the three–dimensional form characteristic of the early comb found in beehives. It contains three axes, denoted x (major axis), y (middle axis) and z (minor axis). Ellipses are two–dimensional and have two perpendicular axes about which the ellipse is symmetric. These axes intersect in the middle of the ellipse. The major axis is the longest distance between antipodal points of the ellipse. The smallest distance across the ellipse is the minor axis. Conventionally, the major radius of the ellipse is denoted *a*, whereas the minor radius is denoted *b*. The minor radius is kept constant in this figure. The ratio *a/b* can reflect the extension of the ellipse : ratios without units and having values of 1.50 (A), 1.70 (B), 2.00 (C) and 3.00 (D) are showing increasing vertical extensions.

Two main procedures were employed for the analysis of the honeycombs. First, natural and intact honeycombs presenting evident elliptical forms were sampled for measurements using a mechanical precision caliper. The same combs were photographed using a digital camera (Panasonic DMC–TZ20 ; 3648 by 2736 pixels) followed by the analysis of the ellipses using the POWERPOINT software (Microsoft Corporation, Redmont, USA). For this, computer–generated ellipses were superimposed to each elliptical honeycomb in the picture, and the peculiar sizes (major and minor axes ; see [21]) were obtained using 200% magnification through the format calculating routine of the computer program. These numbers are given with two decimals by the computer program. Second, pictures of honeycombs were searched in both the scientific or beekeeping literature and on the internet by using a mixture of several specific keywords (such as : *Apis mellifera*, *dorsata, cerana*, honeycombs, colony, wax, frames, comb building & construction, foundationless frames). The digital images were used as described above, or the pictures from books were scanned using a regular printer scanner. The sizes of the elliptical honeycombs were determined electronically as described above. In order to minimize distortions in the measurements, elliptical honeycombs showing both their major and minor axes in their maximal sizes were considered. Appropriate sampling procedures were employed as objectively as possible, except for the pictures to be chosen from the occasions when the honeycombs were presenting their major outlines to the observer, i.e., when they were not observed obliquely.

The statistics reported here are the mean ± 1 standard deviation. To test for the equality of two means, the unilateral Student t-test was employed with a level of significance $\alpha = 0.01$. In order to asses, whether a single extreme value could be removed from each set of data, the Dixon test was employed, with a 5% unidirectional risk. After the removal of one extreme value in each set, the normality test from Shapiro-Wilk was employed, with a level of significance of 0.01.

Results

Natural honeycombs that were built in hives containing 2 to 3 foundationless frames were found to possess evidence for a two–dimensional elliptical structure in their early stage of construction. Initial measurements of these honeycombs were performed using a mechanical caliper. This procedure is providing accurate values for the length of the middle axis of the ellipsoid, but not for its major axis, since the natural honeycomb does present a striction line and a gorge on its upper side, which is not allowing sufficiently accurate measurements. Therefore, this procedure of measurements was abandoned.

In order to obtain more accurate values with these measures, ellipses allowing the measurements of both the major and minor axes of the elliptical honeycomb were electronically drawn on the respective two–dimensional pictures. This procedure was performed with honeycombs originating from my own apiary (Figure 2), from the beekeeping and scientific literature (Figure 3) and from websites in the internet (Figure 4).

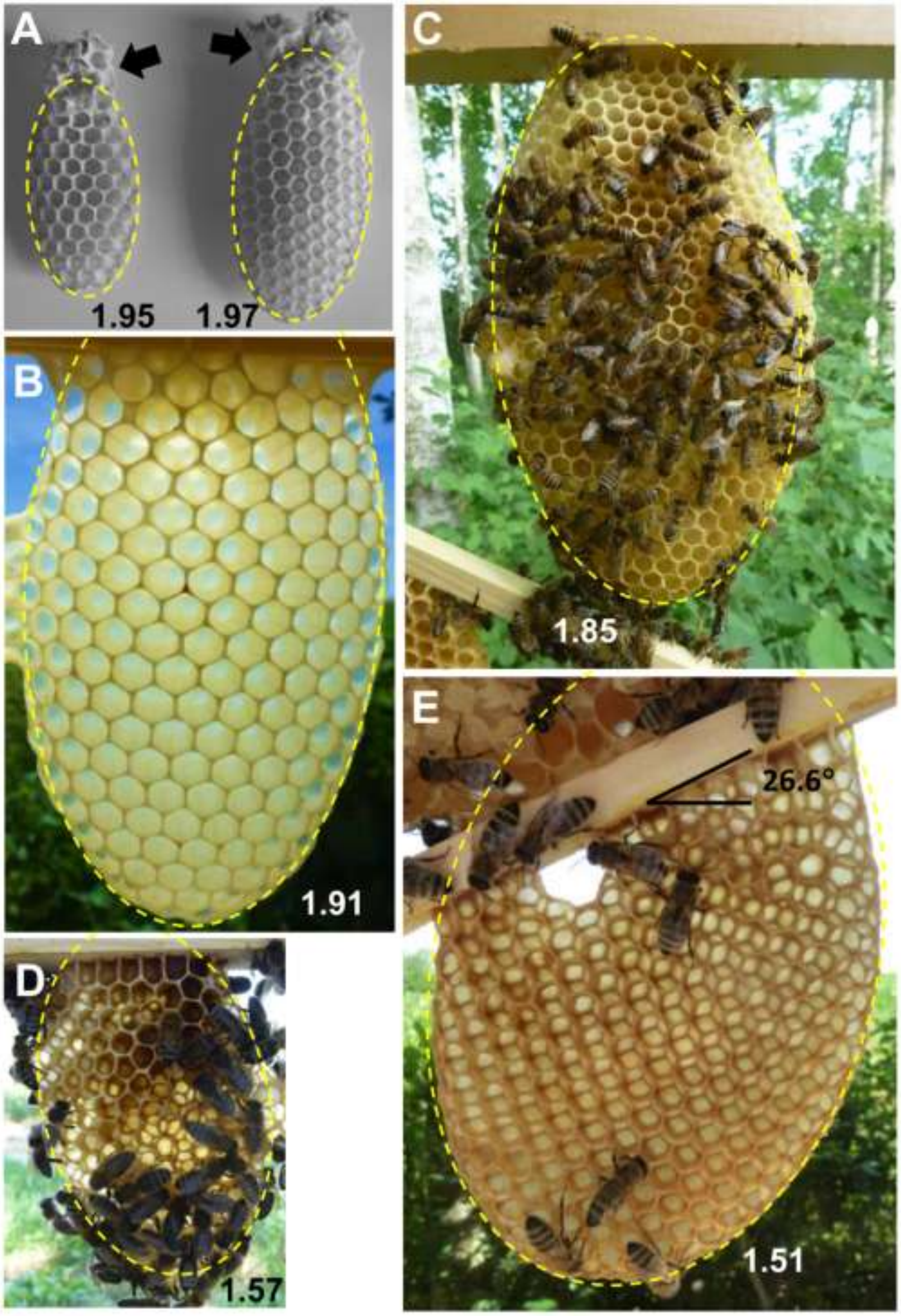

Figure 2. Honeycombs found in hives containing *Apis mellifera carnica* and foundationless frames. (A) Two honeycombs found under the inner cover of the hive. The striction line and the gorge are indicated by an arrow. (B) Honeycomb built in a 10 cm x 12 cm long frame employed for comb honey production. (C), (D), (E) Honeycomb built in a foundationless frame, which has an oblique wooden bar making a plane angle of $\alpha = 26.6°$ with the horizontal bars. All honeycombs were in the brood box, except in (B) in the honey super. The ratio *a/b* (length of the major axis divided by the length of the minor axis) is indicated for each ellipse by a number.

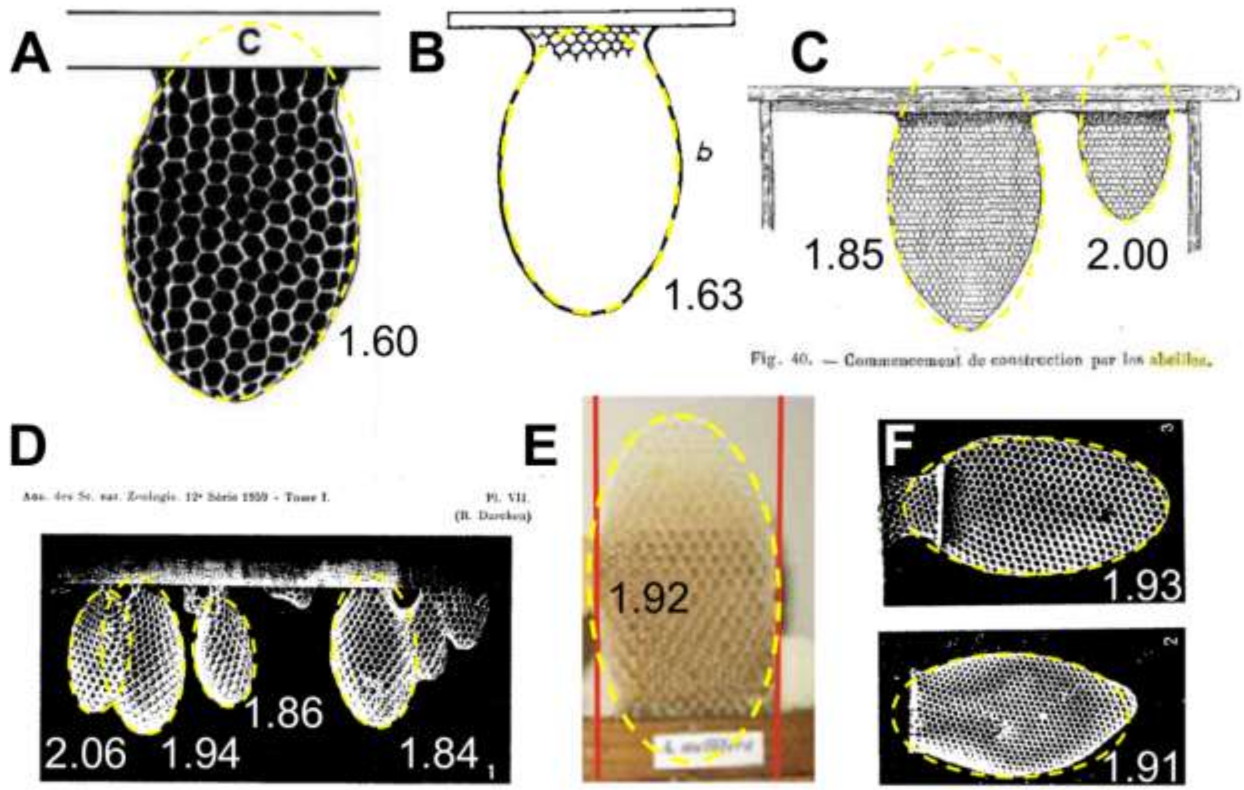

Figure 3. Honeycombs found in the scientific and beekeeping literature. (A) *Apis mellifera capensis* honeycomb ([22] (© EDP Sciences, Les Ulis, France). (B), (D), (F) *Apis mellifera ligustica* honeycombs [13] (© Bernadette Darchen). (C) Early combs from honeybees [23] (Bibliothèque Nationale de France : public domain). (E) : [24] (© Springer–Verlag Berlin Heidelberg). The ratio a/b (length of the major axis divided by the length of the minor axis) is indicated for each ellipse by a number.

This revealed that two kinds of honeycomb ellipses could be found, according to the values of the ratio provided by dividing the length of the long axis by the length of the short axis of the ellipse. These values were found to range either consistently between 1.84 and 2.18 (except for one measure, 2.84 ; n = 28 allowing the generation of a first set of values with a mean 2.0046), or between 1.51 and 1.63 (except for one measure, 1.40 ; n = 16 for the generation of a second set of values with a mean 1.5706). These two sets with all their respective values were further compared together with statistical analyses that are summarized in the Table 1. Using the Student t-test, the t value was $t = 1.42 \times 10^{-13}$. There is 99% probability that the full first set of values falls between the range of 1.9064 and 2.1029 ($P < 0.01$). There is 99% probability that the second full set of values is comprised between 1.5270 and 1.6142 ($P < 0.01$). Finally, the removal of the extreme value 2.84 in the first set of values, and the removal of the minimal value 1.4 in the second set of values, was acceptable for each set of data with a 5% unidirectional risk, as revealed by using the statistical Dixon test. The Shapiro-Wilk normality test performed on both new sets of values revealed that these values were normally distributed (p-value of 0.372 in the first set of values, n = 27 ; p-value of 0.172 in the second set of values, n = 15). When the full set of data is considered, the mean is equal to 1.847, and the confidence interval (alpha=0.05 ; standard deviation of 0.02 ; n=44) is equal to 0.392. Therefore, nothing relevant can be said about the values considered as a whole, thus justufying the generation of two statistically relevant sets of values.

Discussion

To my knowledge, there is no systematic mathematical analysis of the ellipses and the ellipsoids formed by honeycombs in their early stages of construction that has been published in the scientific literature, yet. Although the scrutiny of honeycomb pictures in books or electronic sources is providing valuable data, there is unfortunately a lack of reports or scientific articles dealing with the statistical analyses of such ellipsoidal or elliptical honeycombs. Therefore, the comparison of the data presented in this article with the cognate published scientific or beekeeping literature is quite finical.

To date, an abundant and growing scientific literature is focused on the construction of the honeycomb and its hexagonal cells, the nature and production of beeswax, the manipulation of wax by honeybees, the nests and nesting, the self–organisation of nest contents, the wax gland complex and even the repair of experimentally dislocated cells or combs [13,14,31-34], reviewed in [2,3,35].

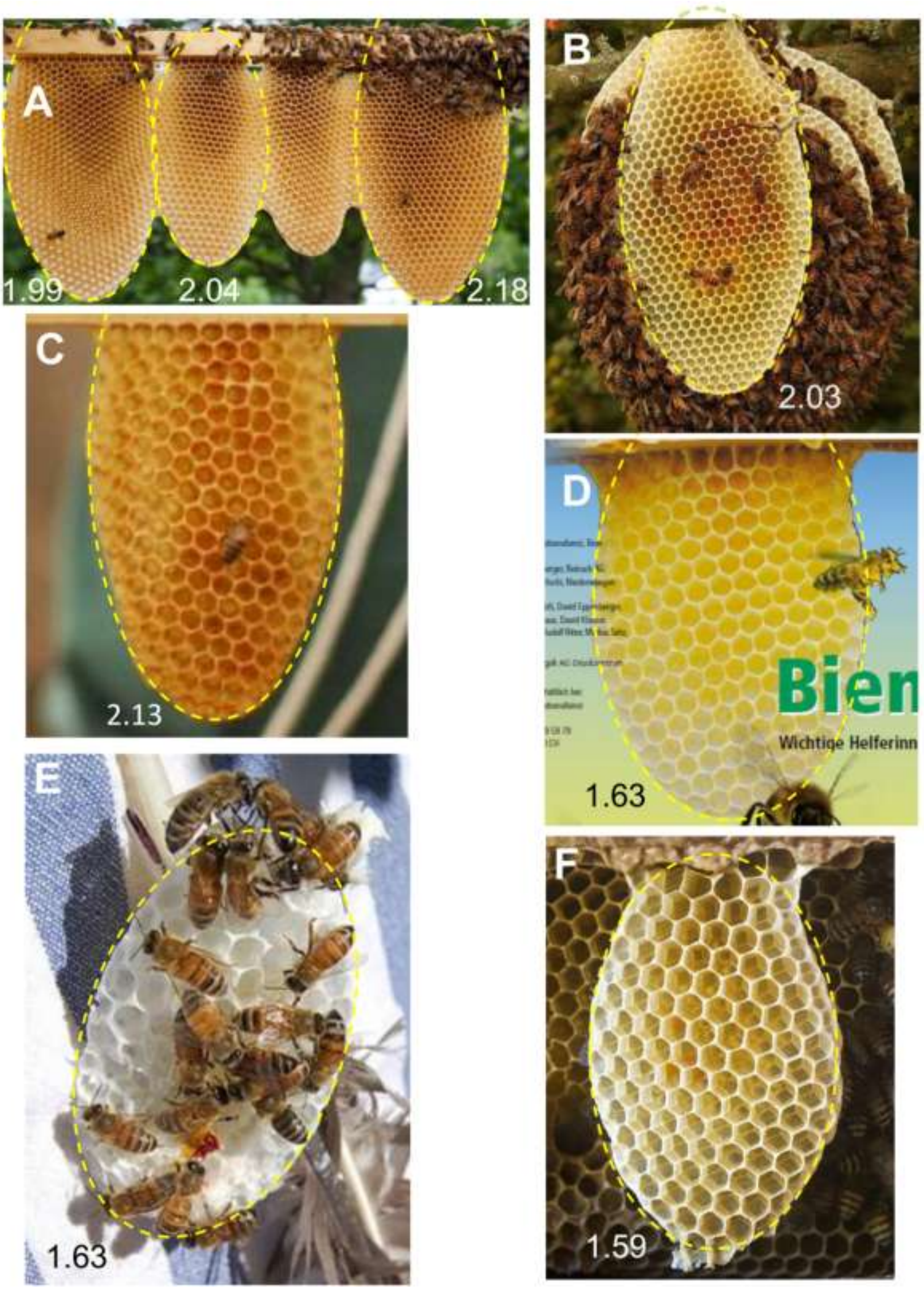

Figure 4. Honeycomb pictures found in internet. (A) Ottawa Honey House [25]. (B) Natural Beekeeping Trust [26] (© The Natural Beekeeping Trust). (C) [27] Backyard Ecosystem (Photographer John Castro. Beekeeper Kevin Murphy and bees in urban Denver CO. Photo used courtesy of BackyardEcosystem.com). (D) Landwirtschaftlicher Informationsdienst [28] (© Landwirtschaftlicher Informationsdienst, LID). (E) Who's Robb ? [29] (© Lisa & Robb). (F) Chitra Katha Pvt. [30] (© Sidharto Rao). The ratio *a/b* (length of the major axis divided by the length of the minor axis) is indicated for each ellipse by a number.

Table 1. Two families of honeycombs according to their elliptical ratio.

Ratio[†]	**1.84 to 2.18**	**1.51 to 1.63**
	[§]2.84	1.63
	2.18	1.63
	2.13	1.63
	2.12	1.63
	2.10	1.60
	2.06	1.59
	2.04	1.59
	2.04	1.58
	2.03	1.58
	2.01	1.58
	2.00	1.57
	2.00	1.55
	1.99	1.53
	1.97	1.53
	1.97	1.51
	1.96	[§]1.40
	1.95	
	1.94	
	1.93	
	1.92	
	1.91	
	1.91	
	1.88	
	1.86	
	1.85	
	1.85	
	1.85	
	1.84	
Mean ± SD[‡]	**2.0046 ± 0.1876**	**1.5706 ± 0.0592**
Confidence interval	**0.1965**	**0.0872**
Variance	**0.034**	**0.003**

[†] The ratio a/b (length of the major axis divided by the length of the minor axis) is indicated for each ellipse by a number (for explanations, see the Figure 1).
[§] Extreme values in each data set. Without their extreme values, the first set of values is comprised between 1.84 and 2.18 ; the second set of values is distributed between 1.51 and 1.63.
[‡] Standard deviation. Calculus was performed with the EXCEL software (2010 version).

The ellipsoidal or elliptical forms of honeycombs have been recognised and attested by ancient and more recent authors [2,13,23,36]. My observations suggest that apart from their evident elliptical structure, early stage honeycombs present peculiar intrinsic mathematical properties. When the length of the long axis of the honeycomb ellipse is divided by the length of its minor axis, the resulting ratio is either very close to the number 2.00 or to the number 1.62, with high probabilities. The two rectangles circumscribing these ellipses are revealing the presence of the golden ratio, Φ, with high statistical probabilities.

In the future, similar observations could be obtained and confirmed by other researchers dealing with hives housing honeybees. Will all the early stage honeycombs fit to these observations? What factors are responsible for putative discrepancies? Do honeybee colonies build preferentially honeycombs with one of these two ratios, or both simultaneously, without any preferences? What are the benefits in terms of statics and building behavior ? Obviously to answer such questions satisfactorily is beyond the scope of this article based on the limited data sets available. A more thorough investigation is therefore needed, in which both the measures of the ellipsoid [37] and the ellipses found in honeycombs might be analysed with procedures involving either real three–dimensional printing or electronic two–dimensional pictures. A real–time investigation in the hive would further allow a precise analysis of the development of the honeycomb.

In order to provide scientific explanations for the understanding of the growing process and the design of the elliptically–growing honeycomb, one should refer to two specific models. First, the diffusion–limited aggregation (DLA) is a model in which the irreversible morphogenesis of the object arises with scale invariance [10,38]. It includes so–called reaction–diffusion processes which are an essential basis for processes involved in morphogenesis in biology [39]. The second model, the constructal theory, covers natural phenomena of organization and the occurrence of design and patterns in nature based on the laws of physics [11,40,41]. The constructal theory, as a self-standing principle, is distinct from the Second Law of Thermodynamics. Epistemologically, its inherent method proceeds from the simple to the complex ; philosophically, the constructal theory "assigns the major role to determinism and contributes significantly to the debate on the origin of living systems" [42,43].

Evidence was presented that "honeybees neither have to measure nor construct the highly regular structures of a honeycomb, and that the observed pattern of combs can be parsimoniously explained by wax flowing in liquid equilibrium. The comb structure is a result of a thermoplastic wax reaching a liquid equilibrium", and that "these interpretations would eliminate the need for bees to perform any mathematical calculations or complex measurements of length and angles" [6]. On the other hand, therefore, we should ask ourselves, whether honeybees do indeed perform calculations involving the golden ratio in order to generate the elliptical honeycombs. In other words : if the honeybees do not perform these calculations, how are such calculations performed ? This provocative question encompasses the mathematical, the philosophical, the teleological and/or the spiritual relevance or the ubiquitous presence of the golden ratio in nature [44].

Conclusion

According to the philosopher Immanuel Kant, artworks are made by rational agents : "*For though we like to call the product that bees make (the regularly constructed honeycombs) a work of art, we do so only by virtue of an analogy with art; for as soon as we recall that their labor is not based on any rational deliberation on their part, we say at once that the product is a product of their nature (namely, of instinct)*" [45]. In the years 1850s, Charles Darwin explained the evolution of the honeybee's comb-building abilities, which was essential for the generation of his theory of natural selection. In order to explain the hexagonal geometry of the bee cells, he personally performed experiments and wrote many letters, the latter being extensively reviewed elsewhere [46]. However, none of these two philosopher or naturalist studied the global elliptical nature of the honeycomb considered as a whole.

The construction of the early honeycomb by honeybees is not a random process, but is following mathematical rules involving the golden ratio. When this honeycomb is presenting an ellipse circumscribed in the long square shown in Figure 5, it reveals a full set of proportions that are related to the golden ratio, as presented in full details elsewhere [47]. When the honeycomb ellipse is circumscribed in the golden rectangle, other interesting properties, also involving the golden ratio, do emerge [48-50].

The study of ellipses, especially the golden ellipse and the long square ellipse, is the *parent pauvre* in the observation of the natural world, since mostly spirals, exponentials, polyhedra, pentads, and helicoïdal or symmetrical patterns have been described [51-56]. Mathematics plays a central role in our current scientific picture of the world. The mathematical explanations in the natural sciences, the explanatory role of mathematics in science and the philosophical relevance of mathematical explanations in science have been reviewed elsewhere :

How the connection between mathematics and the world is to be accounted for remains one of the most challenging problems in philosophy of science, philosophy of mathematics, and general philosophy. A very important aspect of this problem is that of accounting for the explanatory role mathematics seems to play in the account of physical phenomena. [57] The golden ratio is an intrinsic and ubiquitous aspect in such physical phenomena [44].

We should ask ourselves, whether the early phases in the construction of the honeycomb is indeed reflecting the fractal geometry of nature [58]. Fractal structures are abundant in living organisms, ranging from the genetic level, to tissues, organs, organisms and population levels [59]. Moreover, both morphological fractals and temporal fractal structures have been shown to be present within organisms [58]. The growth of the fractal structures has been extensively reviewed not solely for the mineral but also for the animal kingdom [58,60]. Fractal objects do present extremely rich variety of possible realizations of various geometrical objects, to which ellipses might be comprised.

Therefore, the study of the early elliptical honeycomb might thus provide one more note into the hearing of the symphony of life. The fact that the ratios of the honeycomb ellipses do not fit exactly with the values 2.000 or 1.618 can be explained by the observation that in the biological world, living systems are always only approaching their exact mathematical model [53].

The removal of one extreme value in both sets of data is supporting the hypothesis that these two extreme values might belong to a third set of values, namely ellipses circumscribed by rectangles having a ratio which is a multiple of the square root of 2, approximately. Therefore, it is hypothesized that the building of honeycombs by honeybees is following mathematical rules involving irrational numbers, namely the *golden ratio* and possibly the square root of 2. Further observations around the world will substantiate and confirm these early findings.

Might elliptical honeycombs with geometries close to the *golden ratio* reflect a relatively healthy hive ? Today, a widespread colony collapse disorder is affecting hives worldwide. As written elsewhere by a poet, novelist and nature writer, we have to "create less mechanistic stories about *A. mellifera*" ; in this "alert to the plight of pollinators, writers and artists have begun retelling the bee's story"[61].

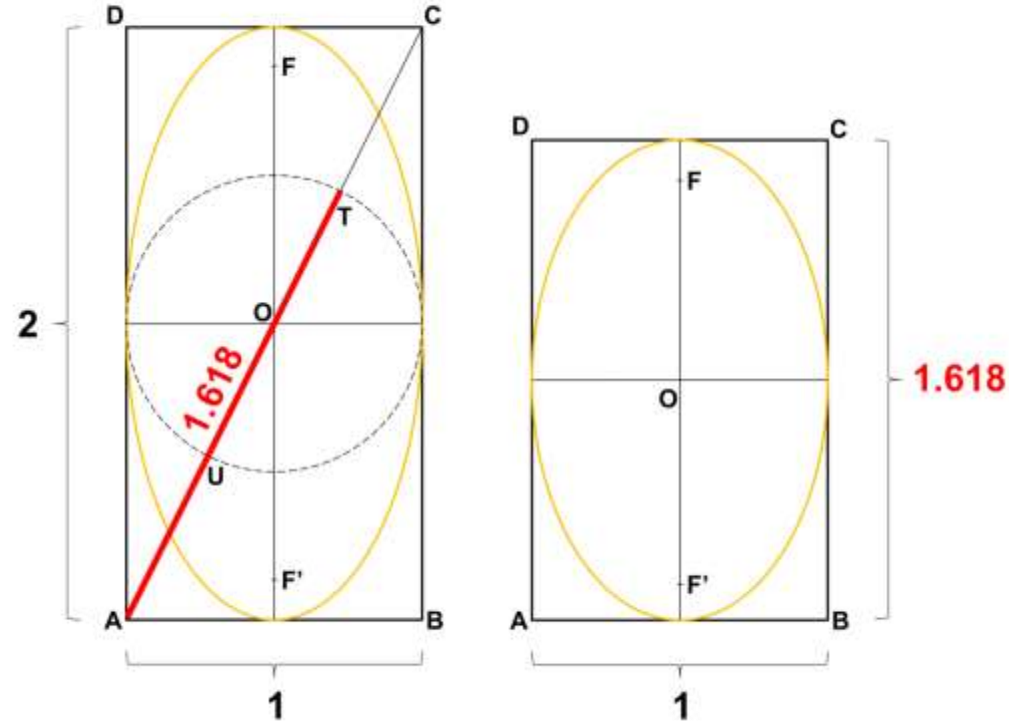

Figure 5. The double square ellipse and the golden ellipse in honeycombs. Left : a rectangle ABCD having its length AD twice longer as its width AB is called the double square. O is the centre of the rectangle. Numbers refer to the different lengths in arbitrary units. The length of the diagonal AC has the value √5. A circle with its centre in O and with a diameter equal to AB is intersecting the diagonal AC in T and U. The distance between the points A and T is equal to the golden ratio Φ = 1.6180339887, approximately (shown in red colour). The distance between the points A and U is equal to Φ–1 = 0.6180339887, approximately. F and F' are the two foci of the ellipse circumscribed in the ABCD rectangle. This ellipse has a ratio (division of the length of the long axis by the length of the short axis of the ellipse) of 2. Its surface is twice the surface of the inner circle. This kind of ellipse is the one that is found in early honeycombs. Right : the golden ellipse comprised within the golden rectangle [48]. An ellipse that would be circumscribed in the golden rectangle having a distance between A and D of 1.6180339887 approximately and a width of 1 is called the golden ellipse. This kind of ellipse is the second one that is found in early honeycombs.

The beehive itself is also, figuratively, a microcosm of the biosphere, a concise and comforting poetic image for the architectonics of ecology. Built out of the living substance of bee bodies, the combs of the hive evoke, in their intricate cell-structure, the architecture of niches that characterizes the biosphere [62]. This architecture is the revelation and the representation of an ancient knowledge, called gnomonicity, related to the word gnomon [63]. The word gnomon was originally given by the Hero of Alexandria, such as "the form that, when added to some form, results to a new form similar to the original". This is thus "a portion of a figure which has been added to another figure so that the whole is of the same shape as the smaller figure" [64]. The addition of successive geometric gnomons to a figure does not alter its proportions, as it is the case for the early honeycomb (Figure 6). The mathematical and scientific explanation of the gnomonicity is thus avoiding the pitfalls of the widespread and unfortunate golden ratio mysticism.

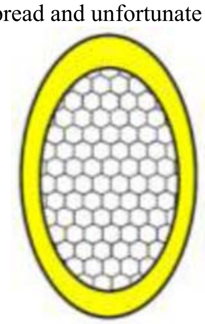

Figure 6: The gnomonic growth of the ideally symmetrical early honeycomb. The elliptical gnomon is shown in yellow. It is resembling an "*elliptical donut*".

Acknowledgements
This research was performed under the full responsibility of the main author (D.F.) I thank Prof. Harald Berresheim for helpful comments and for the critical reading of the manuscript, Lawrence Michel for his advice in the statistical analyses, and Léonard Ribordy for his comment regarding the golden ellipse. This work is dedicated to the memory of the late mathematician Midhat J. Gazalé.

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